

Gravitational waves and electrodynamics: New perspectives

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Given the recent direct measurement of gravitational waves (GWs) by the LIGO-VIRGO collaboration, the coupling between electromagnetic fields and gravity have a special relevance since it opens new perspectives for future GW detectors and also potentially provide information on the physics of highly energetic GW sources. We explore such couplings using the field equations of electrodynamics on (pseudo) Riemann manifolds and apply it to the background of a GW, seen as a linear perturbation of Minkowski geometry. Electric and magnetic oscillations are induced that propagate as electromagnetic waves and contain information about the GW which generates them. We also briefly show the generation of charge density fluctuations induced by GW and the implications for astrophysics.

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I. INTRODUCTION

We have very recently celebrated the first direct measurement of gravitational waves (GWs) by the LIGO-VIRGO collaboration [1], and ESA's LISA-Pathfinder [2] science mission officially started on March, 2016. The waves that were measured by two detectors independently, beautifully match the expected signal following a black hole binary merger, allowing the estimation of physical and kinematical properties of these black holes. This is the expected celebration which not only confirms the existence of these waves and reinforces Einstein's General Relativity (GR) theory of gravity, but it also marks the very birth of GW astronomy. Simultaneously, it gives an indirect observation of black holes and the dynamics of black hole merging in binaries. The measurement was done using laser interferometry, and other methods such as pulsar timing arrays [3] will most probably provide positive detections in the near future.

However, it is crucial to keep investigating different routes towards GW measurements (see [3–7]) and one such route lies at the very heart of this work. Instead of using test masses and measuring the minute changes of their relative distances, as is done in Laser interferometry (used in LIGO, VIRGO, GEO600, TAMA300 and will be used in KAGRA, LIGOIndia and LISA), we can explore the effects of GW on electromagnetic fields. For this purpose, one needs to compute the electromagnetic field equations on the spacetime background of a GW perturbation. This might not only provide models and simulations which can test the viability of such GW-electromagnetic detectors, but it might also contribute to a deeper understanding of the physical properties of astrophysical and cosmological sources of GW, since these waves interact with the electromagnetic fields and plasmas which are expected to be common in many highly

energetic GW sources (see [8]). Thus, an essential aim, in this work, will be to carefully explore the effects of GWs on electromagnetic fields.

Before approaching the issue of GW effects on electromagnetic fields, let us mention very briefly other possible routes in the quest for GW measurements. Recall that linearized gravity is also the context in which gravitoelectric and gravitomagnetic fields can be defined [9], which are present in metrics with time-space components. Similarly, as we will see, the (\times) polarization of GWs is related to space-space off-diagonal metric components, which therefore resemble gravitomagnetism. This analogy might provide a motivation to explore the dynamical effects of GW on gyroscopes. In fact, an analogy with gravitomagnetism brings interesting perspectives regarding physical interpretations, since the analogies with electromagnetism might be explored. In particular, gravitomagnetic effects on gyroscopes are known to be fully analogous to magnetic effects on dipoles. Now, in the case of gravitational waves these analogous (off-diagonal) effects on gyroscopes will, in general, be time dependent.

The tiny gravitomagnetic effect on gyroscopes due to Earth's rotation, was successfully measured during the Gravity Probe B experiment [10], where the extremely small geodetic and Lens-Thirring (gravitomagnetic) deviations of the gyro's axis were measured with the help of Super Conducting Quantum Interference Devices (SQUIDS). Analogous (time varying) gravitoelectromagnetic effects on gyroscopes due to the passage of GWs, might be measured with SQUIDS. On the other hand, rotating superconducting matter seems to generate anomalous (stronger) gravitomagnetic fields (anomalous gravitomagnetic London moment) [11, 12] so, if these results are robustly confirmed then superconductivity and superfluidity might somehow amplify gravitational phenomena. This hypothesis deserves further theoretical and experimental research as it could contribute for GW detectors.

Another promising route comes from the study of the coupling between electromagnetic fields and gravity, the

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topic of our concern in the present work. Are there measurable effects on electric and magnetic fields during the passage of a GW? Could these be used in practice to study the physics of GWs production from astrophysical sources, or applied to GW detection? Although very important work has been done in the past (see for example [8, 13]), it seems reasonable to say that these routes are far from being fully explored.

Regarding electromagnetic waves, it has been shown that gravitational waves have an important effect on the polarization of light [5]. On the other hand, lensing has been gradually more and more relevant in observational astrophysics and cosmology and it seems undoubtedly relevant to study the effects of GWs (from different types of sources) on lensing, since a GW should in principle distort any lensed image. Could lensing provide a natural amplification of the gravitational perturbation signal due to the coupling between gravity and light? These topics need careful analysis for a better understanding of the possible routes (within the reach of present technology) for gravity wave astronomy and its applications to astrophysics and cosmology.

This paper is outlined in the following manner: In Section II, we briefly review the foundations of electrodynamics and spacetime geometry, and present the basic equations that will be used throughout this work. In Section III, we explore the coupling between electromagnetic fields and gravitational waves. In Section IV, we discuss our results and conclude.

II. ELECTRODYNAMICS: GENERAL FORMALISM

In this section, for self-consistency and self-completeness, we briefly present the general formalism that will be applied throughout the analysis. We refer the reader to [14, 15] for a detailed description of the general formalism of electromagnetism in a 4-dimensional spacetime manifold with a pseudo-Riemannian geometry, in the presence of a $(+ - - -)$ signature.

Recall that in the pre-metric formalism of electrodynamics, the charge conservation provides the inhomogeneous field equations while the magnetic flux conservation is given in the homogeneous equations [16–20]. The

field equations are then given by

$$d\mathbf{F} = 0, \quad d\mathbf{G} = \mathbf{J}. \quad (2.1)$$

Note that these are general, coordinate free, covariant equations and there is no need for an affine or metric structure of spacetime. \mathbf{J} is the charge current density 3-form; \mathbf{G} is the 2-form representing the electromagnetic excitation, \mathbf{F} is the Faraday 2-form, so that $\mathbf{F} = d\mathbf{A}$, where \mathbf{A} is the electromagnetic potential 1-form; the operator d stands for exterior derivative.

The constitutive relations (assumed to be linear, local, homogeneous and isotropic) between \mathbf{G} and \mathbf{F} , given by

$$\mathbf{G} = \mu_0^{-1} \star \mathbf{F}, \quad (2.2)$$

provide the metric structure; μ_0 is the magnetic permeability of vacuum and \star is the Hodge star operator mapping k -forms to $(n-k)$ -forms, with n the dimension of the spacetime manifold. On these foundations, the electromagnetic field equations on the background of a general (pseudo) Riemann spacetime manifold are expressed by

$$\partial_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) F^{\mu\nu} = \mu_0 j^\nu, \quad \partial_{[\alpha} F_{\beta\gamma]} = 0, \quad (2.3)$$

where, we have used in the inhomogeneous equations the general expression for the divergence of anti-symmetric tensors in pseudo-Riemann geometry, $\nabla_\mu \Theta^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \Theta^{\mu\nu})$. The homogeneous equations are independent from the metric (and connection) due to the torsionless character of Riemann geometry.

We introduce the definitions

$$F_{0k} = \frac{1}{c} \partial_t A_k - \partial_k A_0 \equiv \frac{E_k}{c}, \quad (2.4)$$

$$F_{jk} = \partial_j A_k - \partial_k A_j \equiv -B_{jk}, \quad (2.5)$$

where $B_{jk} = -\epsilon_{ijk} B^i$ and ϵ_{ijk} is the totally antisymmetric 3-dimensional Levi-Civita (pseudo) tensor. Then, the homogeneous equations are the usual Faraday and magnetic Gauss laws

$$\partial_t B^i + \epsilon^{ijk} \partial_j E_k = 0, \quad \partial_j B^j = 0, \quad (2.6)$$

while the inhomogeneous equations can be separated into the generalized Gauss and Maxwell-Ampère laws. These are, respectively

$$\begin{aligned} \partial_k E_j (g^{0k} g^{j0} - g^{jk} g^{00}) + E_j \left[\partial_k (g^{0k} g^{j0} - g^{jk} g^{00}) + \frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g}) (g^{0k} g^{j0} - g^{jk} g^{00}) \right] \\ - \partial_\mu B^k c g^{m\mu} g^{n0} \epsilon_{kmn} - B^k c \epsilon_{kmn} \left[\partial_\mu (g^{m\mu} g^{n0}) + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) (g^{m\mu} g^{n0}) \right] = \frac{\rho}{\epsilon_0}, \end{aligned} \quad (2.7)$$

$$\begin{aligned} \frac{1}{c} \partial_\mu E_j (g^{0\mu} g^{ji} - g^{j\mu} g^{0i}) + \frac{1}{c} E_j \left[\partial_\mu (g^{0\mu} g^{ji} - g^{j\mu} g^{0i}) + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) (g^{0\mu} g^{ji} - g^{j\mu} g^{0i}) \right] \\ - \epsilon_{kmn} \partial_\mu B^k g^{m\mu} g^{ni} - B^k \epsilon_{kmn} \left[\partial_\mu (g^{m\mu} g^{ni}) + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) (g^{m\mu} g^{ni}) \right] = \mu_0 j^i. \end{aligned} \quad (2.8)$$

One sees clearly, that new electromagnetic phenomena is expected due to the presence of extra electromagnetic couplings induced by spacetime curvature. In particular, the magnetic terms in the Gauss law are only present for non-vanishing off-diagonal time-space components g^{0j} , which in linearized gravity correspond to the gravitomagnetic potentials. These terms are typical of axially symmetric geometries (see [15, 21, 22]).

For diagonal metrics, the inhomogeneous equations, the Gauss and Maxwell Ampère laws, can be recast into the following forms

$$-g^{kk}g^{00}\partial_k E_k + E_k \gamma^k = \frac{\rho}{\varepsilon_0}, \quad (2.9)$$

$$\epsilon_{ijk}g^{ii}g^{jj}\partial_j B^k + \frac{1}{c^2}g^{00}g^{ii}\partial_t E_i + \epsilon_{ijk}\sigma^{jii}B^k + E_i \xi^{ii} = \mu_0 j^i, \quad (2.10)$$

with

$$\gamma^k(\mathbf{x}) \equiv -\left[g^{kk}g^{00}\frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g}) + \partial_k(g^{kk}g^{00})\right], \quad (2.11)$$

and

$$\sigma^{jii}(\mathbf{x}) \equiv g^{jj}g^{ii}\frac{1}{\sqrt{-g}}\partial_j(\sqrt{-g}) + \partial_j(g^{jj}g^{ii}), \quad (2.12)$$

$$\xi^{ii}(\mathbf{x}) \equiv g^{00}g^{ii}\frac{1}{c^2}\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}) + \frac{1}{c^2}\partial_t(g^{00}g^{ii}). \quad (2.13)$$

The Einstein summation convention is applied in Eq. (2.10) only for j and k while the index i is fixed by the right hand side. Also, no contraction is assumed in Eq. (2.11) nor in the expression for σ^{jii} .

New electromagnetic effects induced by the spacetime geometry include an inevitable spatial variability (non-uniformity) of electric fields whenever we have non-vanishing geometric functions γ^k , electromagnetic oscillations (therefore waves) induced by gravitational radiation and also additional electric contributions to Maxwell's displacement current in the generalized Maxwell-Ampère law. This last example becomes clearer by re-writing Eq. (2.10) in the form

$$\epsilon_{ijk}\partial_j \bar{B}^{ijjk} = \mu_0(j^i + j_D^i), \quad (2.14)$$

where $j^i \equiv \sqrt{-g}j^i$ and

$$j_D^i \equiv -\varepsilon_0\sqrt{-g}(g^{00}g^{ii}\partial_t E_i + c^2 E_i \xi^{ii}), \quad (2.15)$$

with the definition

$$\bar{B}^{ijjk} \equiv g^{ii}g^{jj}\sqrt{-g}B^k. \quad (2.16)$$

The functions ξ^{ii} vanish for stationary spacetimes but might have an important contribution for strongly varying gravitational waves (high frequencies), since they depend on the time derivatives of the metric. These are physical, observable effects of spacetime geometry

in electromagnetic fields expressed in terms of the extended Gauss and Maxwell-Ampère laws which help the comparison with the usual inhomogeneous equations in Minkowski spacetime, making clearer the physical interpretations of such effects.

Finally, we review the field equations in terms of the electromagnetic 4-potential which in vacuum are also useful for the issue of electromagnetic waves. From the definition of the Faraday tensor and Eq. (2.3), we get

$$\nabla_\mu \nabla^\mu A^\nu - g^{\lambda\nu} R_{\varepsilon\lambda} A^\varepsilon - \nabla^\nu (\nabla_\mu A^\mu) = \mu_0 j^\nu, \quad (2.17)$$

where $R_{\varepsilon\lambda}$ is the Ricci tensor. Using the expression for the (generalized) Laplacian in pseudo-Riemann manifolds, $\nabla_\mu \nabla^\mu \psi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\lambda}\partial_\lambda\psi)$, and assuming the Generalized Lorenz gauge ($\nabla_\mu A^\mu = 0$) in vacuum, we arrive at

$$\partial_\mu \partial^\mu A^\nu + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\lambda})\partial_\lambda A^\nu - g^{\lambda\nu} R_{\varepsilon\lambda} A^\varepsilon = 0. \quad (2.18)$$

For a diagonal metric, we get

$$\partial_\mu \partial^\mu A^\nu + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\mu})\partial_\mu A^\nu - g^{\nu\nu} R_{\varepsilon\nu} A^\varepsilon = 0, \quad (2.19)$$

with no contraction assumed in ν . In general, and contrary to electromagnetism in Minkowski spacetime, the equations for the components of the electromagnetic 4-potential are coupled even in the (generalized) Lorenz gauge. Notice also that for Ricci-flat spacetimes, the term containing the Ricci tensor vanishes. Naturally, the vacuum solutions of GR are examples of such cases. New electromagnetic phenomena are expected to be measurable, for gravitational fields where the geometric dependent terms in Eq. (2.18) are significant.

This completes the main axiomatic (foundational) formalism of electrodynamics in the background of curved (pseudo-Riemann) spacetime.

III. GRAVITATIONAL WAVES AND ELECTROMAGNETIC FIELDS

As mentioned in the Introduction, GWs have been recently detected by the LIGO team using laser interferometry [1]. Another method that has been carried over the last decade to detect GWs is that of pulsar timing arrays. Nevertheless, it is crucial to keep exploring different routes towards GWs detection and its applications to astrophysics and cosmology. Due to the huge distances in the Cosmos, any GW reaching Earth should have an extremely low amplitude. Therefore, the linearisation of gravity is usually applied which allows to derive the wave equations. It is a perturbative approach which is background dependent and its common to consider a Minkowski background. In any case, the GW can be seen as a manifestation of propagating spacetime geometry perturbations.

In principle, the passage of a GW in a region with electromagnetic fields will have a measurable effect. To compute this we have to consider Maxwell's equations on the perturbed background of a GW. We shall consider a GW as a perturbation of Minkowski spacetime given by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $h_{\alpha\beta} \ll 1$, so that

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 + h_{\alpha\beta} dx^\alpha dx^\beta, \quad (3.1)$$

where the perturbation corresponds to a wave travelling along the z axis, i.e.,

$$ds^2 = c^2 dt^2 - dz^2 - [1 - f_+(z - ct)]dx^2 - [1 + f_+(z - ct)]dy^2 + 2f_\times(z - ct)dx dy, \quad (3.2)$$

and $(+)$ and (\times) refer to the two independent polarizations characteristic of GWs in GR (see for example [23]). This metric is a solution of Einstein's field equations in the linear approximation, in the so-called TT (Transverse-Traceless) Lorenz Gauge [23–26].

A. GW effects on electric and magnetic fields

Consider an electric field in the background of a GW travelling in the z direction. The general expression for Gauss' law (2.7), in vacuum, is now given by

$$0 = [1 - f_+(z, t)]^{-1} \partial_x E_x + [1 + f_+(z, t)]^{-1} \partial_y E_y + \partial_z E_z - \frac{1}{2} f_\times^{-1}(z, t) (\partial_y E_x + \partial_x E_y) + \left[\frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g}) \right] E_z, \quad (3.3)$$

which clearly shows that physical (possibly observable) effects are induced by the propagation of GWs on the electric field.

As for the Maxwell-Ampère law, Eq. (2.8), provides the following relations:

$$\begin{aligned} & \frac{1}{c^2} \left[\frac{1}{2} f_\times^{-1} \partial_t E_y - (1 - f_+)^{-1} \partial_t E_x \right] + E_x \xi^{xx} + E_y \xi^{yx} \\ & - (1 - f_+)^{-1} [(1 + f_+)^{-1} \partial_y B^z - \partial_z B^y] - B^y \sigma^{zxx} \\ & + B^x \sigma^{zyx} - \frac{1}{2} f_\times^{-1} (2f_\times^{-1} \partial_y B^z + \partial_z B^x) = 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned} & \frac{1}{c^2} \left[\frac{1}{2} f_\times^{-1} \partial_t E_x - (1 + f_+)^{-1} \partial_t E_y \right] + E_y \xi^{yy} + E_x \xi^{xy} \\ & - (1 + f_+)^{-1} [(1 - f_+)^{-1} \partial_x B^z - \partial_z B^x] + B^x \sigma^{zyy} \\ & - B^y \sigma^{zxy} + \frac{1}{2} f_\times^{-1} \left(\frac{1}{2} f_\times^{-1} \partial_x B^z + \partial_z B^y \right) = 0, \end{aligned} \quad (3.5)$$

$$\begin{aligned} & -\frac{1}{c^2} \partial_t E_z + E_z \xi^{zz} - \frac{1}{2} f_\times^{-1} (\partial_y B^y - \partial_x B^x) \\ & + [(1 - f_+)^{-1} \partial_x B^y - (1 + f_+)^{-1} \partial_y B^x] = 0, \end{aligned} \quad (3.6)$$

with the non-vanishing geometric coefficients given by

$$\xi^{xx} = \frac{1}{c^2} \frac{4f_\times(f_+ - 1)\partial_t f_\times - (4f_\times^2 + f_+ - 1)\partial_t f_+}{(f_+ - 1)^2(4f_\times^2 + f_+^2 - 1)},$$

$$\xi^{yx} = \xi^{xy} = \frac{1}{c^2} \frac{-(f_+^2 - 1)\partial_t f_\times + f_\times f_+ \partial_t f_+}{2f_\times^2(4f_\times^2 + f_+^2 - 1)},$$

$$\xi^{yy} = \frac{1}{c^2} \frac{-4f_\times(f_+ + 1)\partial_t f_\times + (4f_\times^2 - f_+ - 1)\partial_t f_+}{(f_+ + 1)^2(4f_\times^2 + f_+^2 - 1)},$$

$$\xi^{zz} = -\frac{1}{c^2} \frac{4f_\times(\partial_t f_\times) + f_+(\partial_t f_+)}{4f_\times^2 + f_+^2 - 1},$$

$$\sigma^{zxx} = -\frac{4f_\times(f_+ - 1)\partial_z f_\times - (4f_\times^2 + f_+ - 1)\partial_z f_+}{(f_+ - 1)^2(4f_\times^2 + f_+^2 - 1)},$$

$$\sigma^{zyy} = -\frac{-4f_\times(f_+ + 1)\partial_z f_\times + (4f_\times^2 - f_+ - 1)\partial_z f_+}{(f_+ + 1)^2(4f_\times^2 + f_+^2 - 1)},$$

$$\sigma^{zxy} = \sigma^{zyx} = -\frac{-(f_+^2 - 1)\partial_z f_\times + f_\times f_+ \partial_z f_+}{2f_\times^2(4f_\times^2 + f_+^2 - 1)}.$$

A natural consequence of these laws is the generation of electromagnetic waves induced by gravitational radiation. Initially static electric and magnetic fields become time dependent during the passage of GWs which might be detectable in this way.

Let us start by considering the effects of these GWs on electric fields.

B. Electric field oscillations induced by GWs

We will consider electric fields in the following three simple scenarios.

1. A single electric field along the z axis

In this case, the electric field is aligned with the direction of the GW propagation, so that the Gauss law takes the form

$$\partial_z E_z + E_z \left[\frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g}) \right] = 0, \quad (3.7)$$

where

$$\frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g}) = \frac{4f_\times(\partial_z f_\times) + f_+(\partial_z f_+)}{4f_\times^2 + f_+^2 - 1}, \quad (3.8)$$

which shows that even if in the absence of the GW the field was static and uniform, during the passage of the spacetime disturbance, the field will be time varying and non-uniform, oscillating with the same frequency of the passing GW. The general solution is

$$E_z(z, t) = E_0 \exp \left\{ - \int \left[\frac{4f_\times(\partial_z f_\times) + f_+(\partial_z f_+)}{4f_\times^2 + f_+^2 - 1} \right] dz \right\}, \quad (3.9)$$

where E_0 is considered to be a constant here, but in the most general case it can be a function of x , y and time t . An electric field along the z axis can easily be achieved by charged plane plates, either single or two parallel plates constituting a capacitor. In the absence of GWs the electric field thus produced is uniform for static uniform charge distributions or time variable if there is an alternate current as in the case of a RLC circuit with a variable Voltage signal generator. With the passage of the GW, in general, the electric field is perturbed by both the (+) and (×) modes. It is clear that the GW induces electric oscillations and therefore electromagnetic waves propagating along the same direction of the GW.

As a practical application consider the following GW perturbation

$$f_+(z, t) = a \cos(kz - wt), \quad (3.10)$$

$$f_-(z, t) = b \cos(kz - wt + \alpha). \quad (3.11)$$

In this case, we get the following electric oscillations (which will propagate)

$$E_z(z, t) = \tilde{E}_0 [1 - a^2 \cos^2(kz - wt) - 4b^2 \cos^2(kz - wt + \alpha)]^{-1/2}, \quad (3.12)$$

for $a^2 + 4b^2 \leq 1$, where \tilde{E}_0 is an arbitrary constant and α is the phase difference. These electric oscillations will show distinct features sensitive to the (+) or (×) GW modes. It provides a window for detecting and analysing GW signals directly converted into electromagnetic information.

In order to have an approximate idea on the energy density u^{em} of the resulting electromagnetic wave we can use the usual expression (derived from Maxwell electrodynamics in Minkowski spacetime). We get

$$u^{em} \sim \varepsilon_0 E_z^2(z, t) = \varepsilon_0 \tilde{E}_0^2 [1 - a^2 \cos^2(kz - wt) - 4b^2 \cos^2(kz - wt + \alpha)]^{-1} \quad (3.13)$$

and the energy per unit area and unit time through any surface (with a normal making an angle ϑ with the z axis) is approximately expressed by

$$\|\vec{S}\| \cos \vartheta = \varepsilon_0 c \tilde{E}_0^2 [1 - a^2 \cos^2(kz - wt) - 4b^2 \cos^2(kz - wt + \alpha)]^{-1} \cos \vartheta, \quad (3.14)$$

where \vec{S} is the Poynting vector, and $S \equiv u^{em} c$.

If \tilde{E}_0 is the electric field in the absence of GWs, then the relevant dimensionless quantity to be measured is given by the following expression

$$\frac{|E_z(z, t) - \tilde{E}_0|}{\tilde{E}_0} = \left| [1 - a^2 \cos^2(kz - wt) - 4b^2 \cos^2(kz - wt + \alpha)]^{-1/2} - 1 \right|, \quad (3.15)$$

and in terms of the energy density, we get

$$\frac{|u^{em}(z, t) - u_0^{em}|}{u_0^{em}} = \left| [1 - a^2 \cos^2(kz - wt) - 4b^2 \cos^2(kz - wt + \alpha)]^{-1} - 1 \right|, \quad (3.16)$$

with $u_0^{em} = \varepsilon_0 \tilde{E}_0^2$.

We see that the electromagnetic intensity (energy flux) signal is proportional to the square of the GW amplitudes a and b [for the (+) and (×) modes]. Concerning GW detectors since we are usually dealing with extremely low GW amplitudes reaching the Solar system, the detectors which might have a response proportional to the electric field magnitude or rather to their intensities (to the square of the magnitude), must be extremely sensitive. We emphasize the fact that, in principle, if this electromagnetic wave can be confined in a cavity using very efficient reflectors for the frequency w , then under appropriate (resonant) geometric conditions, the signal can be amplified. This may prove to have very important practical applications for future GW detectors.

2. A single electric field along the x axis

Suppose we have an electric field in the x direction. The electric field could initially be uniform and confined within a plane capacitor. In these conditions, the Gauss law in vacuum becomes

$$[1 - f_+(z, t)]^{-1} \frac{\partial E_x}{\partial x} - (2f_-)^{-1}(z, t) \frac{\partial E_x}{\partial y} = 0. \quad (3.17)$$

A similar expression is obtained if the electric field is aligned with the y axis. If, for instance, only the + mode is present then an initially uniform field is left unperturbed along the x axis, but if both modes are present there will be perturbations on the (x, y) plane.

Assuming a separation of variables $E_x(x, y, z; t) = F_1(x; z; t) F_2(y; z; t)$, where z and t are seen as external parameters and after substituting in the above equation and dividing it by E_x , we obtain

$$2f_- \frac{\partial_x F_1}{F_1} = (1 - f_+) \frac{\partial_y F_2}{F_2}, \quad (3.18)$$

therefore, one arrives at the following solutions

$$F_1(x; z) = C_1 e^{-(1-f_+)x}, \quad F_2(x; z) = C_2 e^{-2f_-y}, \quad (3.19)$$

and finally, the expression

$$E_x(x, y, z, t) = E_{0x} \exp \left\{ - \left[(1 - a \cos(kz - wt)) x + 2b \cos(kz - wt + \alpha) y \right] \right\}, \quad (3.20)$$

which obeys the Gauss law. Again this solution is sensitive to the existence or not of two modes in the GW, to their amplitudes and phase difference.

3. An electric field due to two plane capacitors perpendicular to each other

Let us consider the case where an electric field $\mathbf{E}_1 = (E_x, 0)$ is generated by a plane capacitor oriented along the x axis and a second electric field $\mathbf{E}_2 = (0, E_y)$ is

generated by another similar capacitor oriented along the y axis. In this condition, the resulting electric field in the vacuum between the charged plates, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = (E_x, E_y)$, obeys the equation

$$(1 - f_+)^{-1} \partial_x E_x - (2f_+)^{-1} \partial_y E_x + (1 + f_+)^{-1} \partial_y E_y - (2f_+)^{-1} \partial_x E_y = 0. \quad (3.21)$$

A solution to this equation for the GWs in Eqs. (3.10) and (3.11) is given by

$$E_x = E_{0x} \exp \left\{ - \left[(1 - a \cos(kz - wt)) x + 2b \cos(kz - wt + \alpha) y \right] \right\}, \quad (3.22)$$

$$E_y = E_{0y} \exp \left\{ - \left[2b \cos(kz - wt + \alpha) x + (1 + a \cos(kz - wt)) y \right] \right\}. \quad (3.23)$$

The resulting electric oscillations propagate along the z axis as an electromagnetic wave with non-linear polarization. This wave results from a linear gravitational perturbation of Minkowski spacetime and therefore (in this first order approximation) it can be thought of as an electromagnetic disturbance propagating in Minkowski background. The angle between the resulting electric field and the x axis is then $\Theta(t) \simeq \arctan(E_y/E_x)$, i.e., for $E_{0x} = E_{0y}$

$$\Theta(t) \simeq \arctan \left\{ \exp[-(x + y)a \cos(kz - wt) + (x - y)[1 - 2b \cos(kz - wt + \alpha)]] \right\}. \quad (3.24)$$

Such an oscillating polarization could in principle translate another distinctive signature of the GW that is causing it. Notice that the geometric configuration for the two capacitors here assumed resembles the two perpendicular directions for the laser beams used in LIGO-VIRGO.

Notice that from Eq. (3.17), even assuming both (+) and (\times) modes, it seems that in principle the second term in the Gauss law could be set to zero in the absence of time varying magnetic fields in accordance to Faraday's law. In fact, according to the latter, which is valid before and during the passage of the wave, we get $-\partial_y E_x = \partial_t B^z$, and therefore a static magnetic field would imply that $\partial_y E_x = 0$. Nevertheless one has to be careful since in the presence of gravity our intuition based on electromagnetism in flat spacetime might be misleading. Indeed, due to dynamical spacetime curvature as in GWs, even a static electric field can induce a time varying magnetic field in accordance to the Maxwell-Ampère law.

C. Magnetic field oscillations induced by GWs

One should check if the passage of a GW might inevitably induce a non-vanishing time varying magnetic field, even for an initially static electric field. To illustrate this, consider the case where there is only a (+) polarized

GW (diagonal metric) and an electric field along the x axis. According to Eq. (3.17), the electric field, which is a solution of the Gauss law is independent from x and we can assume it to be intrinsically static. Now, this field will enter in the Maxwell-Ampère law, according to

$$E_x \xi^{xx} + (1 - f_+)^{-1} [(1 + f_+)^{-1} \partial_y B^z - \partial_z B^y] - B^y \sigma^{zxx} = \mu_0 j^x \quad (3.25)$$

$$E_x \xi^{xy} - (1 + f_+)^{-1} [(1 - f_+)^{-1} \partial_x B^z - \partial_z B^x] + B^x \sigma^{zyy} = \mu_0 j^y, \quad (3.26)$$

$$(1 - f_+)^{-1} \partial_x B^y - (1 + f_+)^{-1} \partial_y B^x = \mu_0 j^z. \quad (3.27)$$

The solutions to these equations will give a time varying magnetic field due to the terms containing the derivatives of the metric. This is in complete contrast with the situation of usual electromagnetism in flat spacetime, where for stationary currents and static electric fields, the solutions to the Maxwell-Ampère law gives rise to a static magnetic field. This example shows that, in principle, a time variability in the magnetic field can be induced by the GW and will affect the electric field due to Faraday's law. This system of equations can be explored numerically to compute the resulting magnetic oscillations which could be measured in principle using SQUIDS (Super Conducting Quantum Interference Devices) that are extremely sensitive to small magnetic field changes.

To get a glimpse of the gravitationally induced magnetic field fluctuations, we can consider for simplicity only the (+) GW mode and take the generalized Maxwell-Ampère law in the form of Eq. (2.14). Then we can perform an integration over an amperian closed line coincident to a magnetic field line (in perfect analogy with the method taken in usual electromagnetism to integrate the Maxwell-Ampère law), assuming axial symmetry, around some charge current distribution and some electric flux (Maxwell displacement) current.

The Maxwell displacement current can be entirely due to the passing GW. Indeed, even in the absence of charge currents, the gravitationally induced electric oscillations can in principle generate the magnetic field.

Consider a magnetic field with components B_x and B_y , due to some current along the z axis and the previously obtained electric field along the z axis, given by Eq. (3.9). We obtain in this case

$$\vec{B} = \frac{\mu_0 I_{tot}}{2\pi \sqrt{x^2 + y^2}} (\cos \phi \vec{e}_y - \sin \phi \vec{e}_x), \quad (3.28)$$

where $\vec{B} = (\bar{B}^{zzyy}, \bar{B}^{zzxx})$, $\bar{B}^{ijjk} \equiv g^{ii} g^{jj} \sqrt{-g} B^k$, $I_{tot} = I + I_D$ and $I_D = \int \int j_D^z dx dy$. I is the charge current. We then get the magnetic field components

$$B^x = - \frac{1 + f_+}{\sqrt{1 - f_+^2}} \left(\frac{\mu_0 I}{2\pi \sqrt{x^2 + y^2}} \sin \phi \right), \quad (3.29)$$

and

$$B^y = \frac{1-f_+}{\sqrt{1-f_+^2}} \left(\frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}} \cos \phi \right), \quad (3.30)$$

respectively.

Here, $\phi = \arctan(y/x)$ and in this case $I_D = 0$ since $j_D^z(z, t) = 0$. This can be seen from Eq (2.15), from which we can write the following expression

$$j_D^i(z, t) = -\varepsilon_0 \partial_t (g^{00} g^{ii} \sqrt{-g} E_i). \quad (3.31)$$

Therefore $j_D^z(z, t) = \varepsilon_0 \partial_t (\sqrt{-g} E_z) = 0$ since $E_z = E_{0z}/\sqrt{-g}$, where E_{0z} is time-independent.

We then have the following magnetic field fluctuations

$$B^x = -\frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}} \sin \phi \left[\frac{1+a \cos(kz-wt)}{\sqrt{1-a^2 \cos^2(kz-wt)}} \right], \quad (3.32)$$

$$B^y = \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}} \cos \phi \left[\frac{1-a \cos(kz-wt)}{\sqrt{1-a^2 \cos^2(kz-wt)}} \right]. \quad (3.33)$$

We can easily see that for any point (x, y) fixed on any magnetic field line, the x and y components of the magnetic field will oscillate in time out of phase, such that when one is at its maximum value, the other is at the minimum, and vice-versa. The overall result is that the magnetic field lines will oscillate with the passage of the GW, following the deformations of the spacetime geometry.

Suppose now another example where the magnetic field has components on the yz plane. Consider an electric field along the x axis and also the presence of a current on the same axis. In this case, we have

$$\vec{B} = \frac{\mu_0 I_{tot}}{2\pi\sqrt{y^2+z^2}} \left(\cos \tilde{\phi} \vec{e}_z - \sin \tilde{\phi} \vec{e}_y \right), \quad (3.34)$$

where $\vec{B} = (\bar{B}^{xxzy}, \bar{B}^{xxyy})$. Therefore, the magnetic field components are given by

$$B^y = -\frac{1-f_+}{\sqrt{1-f_+^2}} \left(\frac{\mu_0 I_{tot}}{2\pi\sqrt{y^2+z^2}} \sin \tilde{\phi} \right), \quad (3.35)$$

$$B^z = \sqrt{1-f_+^2} \left(\frac{\mu_0 I_{tot}}{2\pi\sqrt{y^2+z^2}} \cos \tilde{\phi} \right). \quad (3.36)$$

Here, $\tilde{\phi} = \arctan(z/y)$ and in this case $I_D = \int \int j_D^x dy dz$, where from Eq. (3.31), we have

$$j_D^x(x, z, t) = \varepsilon_0 \partial_t \left[(1-f_+) \sqrt{1-f_+^2} E_x \right], \quad (3.37)$$

which, in general, is non-zero, even if E_x were static. The Maxwell displacement current can be entirely induced by

the GW and, consequently, even in the absence of charge currents a non-vanishing and dynamical magnetic field can arise.

Finally, the resulting magnetic oscillations are given by

$$B^y = C^y(y, z) I_{tot} \left[\frac{1-a \cos(kz-wt)}{\sqrt{1-a^2 \cos^2(kz-wt)}} \right], \quad (3.38)$$

$$B^z = C^z(y, z) I_{tot} \sqrt{1-a^2 \cos^2(kz-wt)}, \quad (3.39)$$

where C^y and C^z are taken from Eqs (3.35) and (3.36).

D. Charge density fluctuations induced by GWs

Consider now the effect of the propagating GW on charge distributions, derived from the conservation equation in general (pseudo) Riemann spacetime ($\nabla_\mu j^\mu = 0$). We saw in Ref. [14] that even in the absence of (intrinsic) currents, a non-static spacetime will induce a time variability in the charge density according to

$$\partial_t \rho = -\Gamma_{\mu 0}^\mu \rho c = -\partial_t (\log(\sqrt{-g}) \rho), \quad (3.40)$$

therefore we can write

$$\rho(t) = \rho_0 \sqrt{\frac{g_0}{g(t)}}, \quad (3.41)$$

where ρ_0 is the initial charge density before the passage of the wave and g_0 is the determinant of the initially unperturbed background metric. For the simpler case of GWs travelling along the z direction, seen as disturbances of Minkowski spacetime, we have the simple result

$$\rho(t) = \rho_0 (1 - 4f_x^2 - f_+^2)^{-\frac{1}{2}}. \quad (3.42)$$

As an example, for the harmonic GW modes considered previously, we obtain

$$\rho(z, t) = \rho_0 [1 - 4b^2 \cos^2(kz - wt + \alpha) - a^2 \cos^2(kz - wt)]^{-\frac{1}{2}}. \quad (3.43)$$

Consequently, one naturally predicts charge density fluctuations and, therefore, currents due to the passage of GWs. Such density oscillations propagate along the z direction following the GW penetrating a conducting material medium. This is analogous to Alfvén waves in plasmas, which are density waves induced by magnetic disturbances which propagate along the magnetic field lines. In this case, astrophysical sources of GW such as Gamma Ray Bursts or coalescing binaries that happen to be surrounded by plasmas in accretion disks or stellar atmospheres, might generate similar mass density waves and charge density waves induced by the GW propagation. A more realistic treatment would require the equations of Magneto-Hydrodynamics in the background of a GW (see Ref. [8]). An even more realistic study would

consider the backreaction of the relativistic plasma and electromagnetic fields on the GW properties such as the frequency, amplitude and polarizations, so that the travelling wave after detection could, in principle, contain information about the physical properties of the medium through which it propagated.

The above expression can also indicate another window for GW detection, using conductors in perfect electrostatic equilibrium or superconducting materials at very low temperatures which might reveal very dim electric oscillations with well-defined characteristics, induced by GWs.

E. GW effects on electromagnetic waves

The vacuum equations for the 4-potential in the presence of a background gravity wave can be derived from Eq. (2.18). In terms of the electric and magnetic components of the 4-potential, we have

$$\partial_\mu \partial^\mu \phi + \frac{4f_\times (\partial_t f_\times) + f_+ (\partial_t f_+)}{4f_\times^2 + f_+^2 - 1} \partial_t \phi - \frac{4f_\times (\partial_z f_\times) + f_+ (\partial_z f_+)}{4f_\times^2 + f_+^2 - 1} \partial_z \phi = 0, \quad (3.44)$$

$$\partial_\mu \partial^\mu A^k + \frac{4f_\times (\partial_t f_\times) + f_+ (\partial_t f_+)}{4f_\times^2 + f_+^2 - 1} \partial_t A^k - \frac{4f_\times (\partial_z f_\times) + f_+ (\partial_z f_+)}{4f_\times^2 + f_+^2 - 1} \partial_z A^k = 0, \quad (3.45)$$

respectively.

The resulting expressions simplify significantly if one considers only one of the two possible GW modes. For example, for an electromagnetic wave travelling in the z direction and the harmonic GW in Eq. (3.11) with no (\times) mode, we get the following wave equation for the electric potential

$$\partial_{tt}^2 \phi - \partial_{zz}^2 \phi - \frac{wa^2 \sin(wt - kz) \cos(wt - kz)}{a^2 \cos^2(wt - kz) - 1} \partial_t \phi - \frac{ka^2 \sin(wt - kz) \cos(wt - kz)}{a^2 \cos^2(wt - kz) - 1} \partial_z \phi = 0, \quad (3.46)$$

that can be studied by applying Fourier transformation methods.

In order to study in depth the physical (measurable) effects of the passage of the GW on electromagnetic wave dynamics, one needs to solve these equations and then compute the gauge invariant electric and magnetic fields. We see in the above wave equations, the presence of terms proportional to the first derivatives which are completely absent in the electromagnetic wave equations in flat spacetime (in cartesian coordinates). These terms are always induced by gravitational fields, but in this case the gravitational field is dynamical which represents a much richer electromagnetic wave signal with the signature of the GW (see also [13]). Such signals in the

radio regime might possibly be detectable through methods of Long Base Interferometry, in order to amplify it. Nevertheless, we can see from the expressions above that the extra terms on the electromagnetic wave equations, induced by GWs are proportional to the frequency. Such gravitational effects might become important for sufficiently high frequency GWs. Simulations are required to see the feasibility or not of such methods.

IV. DISCUSSION AND FINAL REMARKS

GW astronomy is an emerging field of science with the potential to revolutionize astrophysics and cosmology. The construction of GW observatories can also effectively boost major technological developments. Given the extremely low GW amplitudes reaching the Solar System, incredibly huge laser interferometers have been built and others are under development in order to reach the required sensitivities. In fact, the biggest of all, e-LISA is expected to be achieved in space possibly through an ESA-NASA collaboration. ESA's LISA-Pathfinder science mission officially started on March 2016 and over the next six months it will conduct hundreds of experiments to pave the way for future space GW observatories, such as eLISA. These huge observatories represent an amazing technological effort. A natural question is the following: Can we amplify the GW signal? One fundamental prediction of the coupling between gravity and electromagnetism is the generation of electromagnetic waves due to gravitational radiation. Therefore, in principle under the appropriate resonant conditions, the electromagnetic signal thus produced can be amplified allowing us to measure GWs, not through the motion of test masses but rather by transferring the GW signal directly into electromagnetic information. This fact might represent an important change in perspective for future ground and space GW detectors.

The fact that GWs can generate electromagnetic waves is of course not evident if one restricts the analysis to the propagation of light rays (in the geometrical optics limit) in curved spacetime. On the other hand, the full Einstein-Maxwell system of equations have to take into account the curved spacetime within Maxwell's equations and also the contribution of the electromagnetic stress-energy tensor to the gravitational field. The first aspect of this coupling was considered in this work, and it is sufficient to show that GWs can be sources of electromagnetic waves. The full gravity-electromagnetic coupling also shows the reverse phenomenon.

In this work, we obtained electric field oscillations fully induced by a GW travelling along the z axis. For simplicity we assumed harmonic GWs. We considered the Gauss law for the cases of an electric field along the z axis, along the x axis and in the space between two plane capacitors perpendicular to each other and to the z axis. In the first case, the solutions in Eqs. (3.9) and (3.12) allowed to make an estimation of the energy flux of the

resulting electromagnetic wave. We also obtained electric field oscillations in the x, y plane. For example, for the electric field between two capacitors in perpendicular configuration, the electric oscillations propagating along the z axis as electromagnetic waves have non-linear polarization. This contrasts with the other cases where the resulting wave was linearly polarized. In all cases, the resulting electromagnetic signal has the signature of the GW that produces it.

In any of these examples, time varying electric fields are generated, which can contribute to the magnetic field via the Maxwell-Ampère law. In particular, they enter the generalized Maxwell displacement current vector density, Eq. (2.15), induced by the GW. This in turn can generate a time varying magnetic field even in the absence of charge currents. Accordingly, GWs also induce magnetic field oscillations, derivable from the Maxwell-Ampère law, Eqs. (3.25)-(3.27). We made an estimation of such an effect by considering the cases of a magnetic field on the (x, y) plane and on the (y, z) plane. In the first case, for example, we see from Eqs. (3.32) and (3.33) that for any point (x, y) fixed on the magnetic field line, the x and y components of the magnetic field will oscillate in time out of phase, such that when one is at its maximum value, the other is at the minimum, and vice-versa. The overall result is that the magnetic field lines will oscillate with the passage of the GW, following the deformations of spacetime geometry.

Concerning GW measurements using electric and magnetic fluctuations induced by GW, since we are usually dealing with extremely low GW amplitudes reaching the Solar System, the detectors must be extremely sensitive. These might have a response proportional to the electric or magnetic field magnitudes or rather to their intensities (proportional to the square of the magnitudes), for example. The changing electric field in Eq. (3.9) inside a capacitor, for instance, would also generate alternate currents in any conductor placed between the capacitor's charged plates. In particular, a diode placed in the appropriate orientation would allow a signal current in a single direction intermittently, following the rhythm of the GW fluctuations. Analogously, small magnetic field changes could in principle be measured with SQUIDS, which are sensitive to extremely small magnetic fields.

We also obtained charge density oscillations induced

by GW. These can propagate as density waves in the case of charged fluids, through which a GW is propagating. This effect deserves to be taken in consideration within more complete magnetohydrodynamical computations, in order to have simulations of the effects of GW in plasmas near the cores of highly energetic GW sources. These plasma environments might occur in different astrophysical sources such as Gamma Ray Bursts and some coalescing binaries.

Regarding electromagnetic waves in the presence of gravity, extra terms appear in the generalized wave equations (see [15]) which deserves further research to get a full analysis of the approximate solutions. Indeed, going beyond the geometrical optics limit, light deflection (null geodesics) and gravitational redshift are not the only effects arising from the coupling between light and gravity. More generally, all electromagnetic waves can experience gravitational effects on the amplitudes, frequencies and polarizations. Besides, as shown in [15], electric and magnetic wave dynamics become coupled due to the non-stationary geometries, as in the presence of GWs.

In general, one expects that GWs induce very rich electromagnetic wave dynamics. We saw that these effects might become more significant for very high frequency GWs as one can see from Eq. (3.46). Moreover, the terms proportional to the first derivatives of the 4-potential have space and time varying coefficients. For the harmonic GWs considered in this work, these coefficients oscillate between positive and negative numbers, a fact that might imply a very distinctive wave modulation pattern of the resulting electromagnetic wave. This hypothesis and its implications requires further investigation as it might provide very rich GW information codified in the electromagnetic spectra of different astrophysical and even cosmological sources.

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